Since the first two equations of Eqs. (1) are independent of B_x, which can be found after u and E_z are determined, we will only study the solutions of u and E_z. Similar to the method used in a previous paper,[2] we use Duhamel's theorem for a system of coupled equations and find that the solutions for any given pressure variation are

$$u = \frac{\cosh M - \cosh M\xi}{\sinh M}\,\frac{M}{\sigma B_0^2}\left[P_s + \frac{\partial}{\partial t}\int_0^t P_1(t')dt'\right] + \frac{h^2}{\rho\nu}\sum_0 \cos\frac{(2n+1)\pi\xi}{2}\,\frac{\partial}{\partial\tau}\int_0^\tau P_1(\tau') \times \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)(\tau-\tau')\right]\times\{A_{2n+1}\times \sin\,[R_{2n+1}(\tau-\tau')] + B_{2n+1}\cos[R_{2n+1}(\tau-\tau')]\}d\tau' \quad (4)$$

$$E_z = \frac{1 - M\coth M}{\sigma B_0}\left[P_s + \frac{\partial}{\partial t}\int_0^t P_1(t')dt'\right] + \frac{1}{\sigma B_0}\times \sum_0 \cos\frac{(2n+1)\pi\xi}{2}\,\frac{\partial}{\partial\tau}\int_0^\tau P_1(\tau')\exp\left[-\frac{(2n+1)^2\pi^2}{8}\times \left(1+\frac{\lambda}{\nu}\right)(\tau-\tau')\right]\left\{\left\langle\left[\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right]A_{2n+1} + R_{2n+1}B_{2n+1}\right\rangle \sin[R_{2n+1}(\tau-\tau')] + \left\langle\left[\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right]B_{2n+1} - R_{2n+1}A_{2n+1}\right\rangle \cos[R_{2n+1}(\tau-\tau')]\right\}d\tau' \quad (5)$$

where

$$\xi = y/h, \quad \tau = \nu t/h^2, \quad M = \sigma B_0^2 h^2/\rho\nu$$

$$R_n = \frac{n\pi}{2}\left[\frac{\lambda}{\nu}M^2 - \frac{n^2\pi^2}{16}\left(1 - \frac{\lambda}{\nu}\right)^2\right]^{1/2}$$

$$A_{2n+1} = \frac{2}{R_{2n+1}}\,\frac{(-1)^n}{(2n+1)\pi}\left[\frac{(1+\lambda/\nu)(2n+1)^2\pi^2}{4M^2 + (2n+1)^2\pi^2}\times M\coth M - 2\right] \quad (6)$$

$$B_{2n+1} = \frac{(-1)^{n+1}}{(2n+1)\pi[4M^2 + (2n+1)^2\pi^2]}\,16M\coth M$$

Special Cases

1. *Sudden Start*

We consider the problem that the fluid is initially at rest and is set in motion by a sudden application of a constant axial pressure gradient. This readily implies that $P_s = 0$ and $P_1 =$ const. Performing the integrations, we find

$$u = \frac{P_1 M}{\sigma B_0^2}\left[\frac{\cosh M - \cosh M\xi}{\sinh M}\right] + \frac{P_1 h^2}{\rho\nu}\times \sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right][A_{2n+1}\sin R_{2n+1}\tau + B_{2n+1}\cos R_{2n+1}\tau]\cos\frac{(2n+1)\pi\xi}{2} \quad (7)$$

$$E_z = \frac{P_1}{\sigma B_0}(1 - M\coth M) + \frac{P_1}{\sigma B_0}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\times \left(1+\frac{\lambda}{\nu}\right)\tau\right]\left\{\left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)A_{2n+1} + R_{2n+1}B_{2n+1}\right]\sin R_{2n+1}\tau + \left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)B_{2n+1} - R_{2n+1}A_{2n+1}\right]\cos R_{2n+1}\tau\right\}\cos\frac{(2n+1)\pi\xi}{2} \quad (8)$$

As $t \to \infty$, we may recover the steady Hartmann problem.[3] Also we may recover the solution in a nonmagnetic field by letting $M \to 0$.

2. *Sudden Removal of the Pressure Gradient*

A steady state is suddenly changed by the removal of the pressure gradient. This states that $P_1 = -P_s$. The solutions become

$$u = -\frac{P_s h^2}{\rho\nu}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right]\times [A_{2n+1}\sin R_{2n+1}\tau + B_{2n+1}\cos R_{2n+1}\tau]\cos\frac{(2n+1)\pi\xi}{2} \quad (9)$$

$$E_z = -\frac{P_s}{\sigma B_0}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right]\times \left\{\left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)A_{2n+1} + R_{2n+1}B_{2n+1}\right]\sin R_{2n+1}\tau + \left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)B_{2n+1} - R_{2n+1}A_{2n+1}\right]\cos R_{2n+1}\tau\right\}\cos\frac{(2n+1)\pi}{2}\xi \quad (10)$$

To illustrate the oscillatory nature of the problem, we present some numerical results of the case of sudden start with magnetic Prandtl number $\lambda/\nu = 1$. Figure 1 shows the velocity profiles of different Hartmann numbers in dimensionless velocity $(\rho\nu/P_1h^2)u$ at the dimensionless time $\tau = \nu t/h^2 = 0.1$. Figure 2 gives the time history of the velocity magnitudes at the center of the channel and that of the average velocity.

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The Use of Macauley's Brackets in the Analysis of Laterally Loaded Struts and Tie-Bars

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Nomenclature

a = x-coordinate locating lateral load
C_1, C_2 = integration constants
EI = flexural rigidity
k = $(P/EI)^{1/2}$
L = span
M = bending moment

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Table 1 Macauley terms due to lateral loads on axially loaded struts and tie-bars

Type of lateral load	Term to be added to basic bending-moment equation	Expression to be added to basic deflection equation: Strut	Expression to be added to basic deflection equation: Tie-bar
Load W concentrated at $x = a$	$-W\langle x - a\rangle$	$\frac{W}{kP}\sin k\langle x - a\rangle - \frac{W}{P}\langle x - a\rangle$	$\frac{W}{P}\langle x - a\rangle - \frac{W}{kP}\sinh k\langle x - a\rangle$
Uniform load of intensity w per unit length from $x = a$ to $x = L$	$-\frac{w}{2}\langle x - a\rangle^2$	$\frac{2w}{k^2P}\sin^2 \frac{1}{2}k\langle x - a\rangle - \frac{w}{2P}\langle z - a\rangle^2$	$\frac{w}{2P}\langle x - a\rangle^2 - \frac{2w}{k^2P}\sinh^2 \frac{1}{2}k\langle x - a\rangle$
Clockwise couple M_0 applied in plane of bending at $x = a$	$M_0\langle x - a\rangle^0$	$\frac{2M_0}{P}\sin^2 \frac{1}{2}k\langle x - a\rangle$	$\frac{2M_0}{P}\sinh^2 \frac{1}{2}k\langle x - a\rangle$

M_0 = applied bending couple
M_1 = fixing moment at $x = 0$
P = axial end load
R_1 = lateral reaction at $x = 0$
W = concentrated lateral load
w = intensity of uniformly distributed load
x,y = coordinates of deflection curve

A RECENT note[1] shows that the integration of the discontinuous expressions arising in beam deflection problems may be simplified by the use of Macauley brackets. The method can be extended to bars which are subjected to combinations of axial and lateral loads.

Macauley brackets take the form $\langle x - a\rangle$ and are given the property that terms containing them are omitted when the expression inside these brackets becomes negative—that is when $x < a$. Thus,

$$f\langle x - a\rangle = \begin{cases} 0 & \text{for } x < a \\ f(x - a) & \text{for } x > a \end{cases}$$

Struts with Hinged Ends

In the notation of Fig. 1 the bending-moment equation for $x < a$ may be written

$$M = -Py + R_1x \tag{1}$$

and this will be termed the basic bending-moment equation. The substitution of Eq. (1) in the relationship $EI(d^2y/dx^2) = M$ leads to the basic deflection equation,

$$y = C_1 \sin kx + C_2 \cos kx + R_1x/P \tag{2}$$

where $k^2 = P/EI$.

If we extend Eq. (1) with a Macauley-bracket term which allows for the lateral load W we obtain the following equation which represents the bending moment throughout the strut:

$$M = -Py + R_1x - W\langle x - a\rangle \tag{3}$$

A single equation representing the deflection curve of the entire strut must satisfy Eq. (3). Furthermore, the deflection y and slope dy/dx must be continuous at $x = a$ with unchanging values of C_1 and C_2. These conditions are satisfied by adding to Eq. (2) the expression

$$(W/kP)\sin k\langle x - a\rangle - (W/P)\langle x - a\rangle$$

Table 1 shows the expressions to be added to the basic bending-moment and deflection equations for the types of lateral load occurring frequently.

Any number of lateral loads and applied couples can be accommodated in a single deflection equation which requires only two constants of integration. This contrasts with the classical solution[2] in which the number of equations increases as more lateral loads are applied.

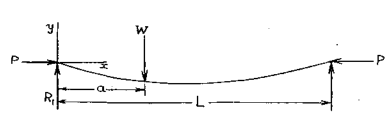

Fig. 1

The reaction R_1 may be obtained from the conditions $M = y = 0$ at $x = L$. The constants C_1 and C_2 are determined from the end conditions $y = 0$ when $x = 0$ and $y = 0$ when $x = L$. The first of these gives $C_2 = 0$.

The term $\langle x - a\rangle^0$ possesses the usual Macauley property. An example of this occurs in Table 1. Thus,

$$M_0\langle x - a\rangle^0 = \begin{cases} 0 \text{ for } x < a \\ M_0(x - a)^0 = M_0 \text{ for } x > a \end{cases}$$

Struts with Built-In Ends

Table 1 is also applicable to struts with direction-fixed ends. Suppose the bending moment at the left-hand end is M_1. The basic bending-moment and deflection equations become

$$M = M_1 - Py + R_1x$$

$$y = C_1 \sin kx + C_2 \cos kx + \frac{M_1}{P} + \frac{R_1 x}{P}$$

The Macauley terms from Table 1 are added and the values of C_1,C_2,M_1, and R_1 are then determined from the conditions $y = dy/dx = 0$ when $x = 0$ and also when $x = L$. In the determination of the expression for slope dy/dx the derivative of the Macauley deflection expression for a concentrated load W (see Table 1) is written

$$(W/P)\cos k\langle x - a\rangle - (W/P)\langle x - a\rangle^0$$

to insure that the resulting function is continuous at $x = a$. However, it is convenient to replace this result by the equivalent single term

$$-(2W/P)\sin^2 {}^1/_2k\langle x - a\rangle$$

The Bars with Lateral Loads

If we change the axial load P from compressive to tensile, the basic bending-moment and deflection equations become:

For hinged ends:

$$M = Py + R_1x$$

$$y = C_1 \sinh kx + C_2 \cosh kx - (R_1x/P)$$

For built-in ends:

$$M = M_1 + Py + R_1x$$

$$y = C_1 \sinh kx + C_2 \cosh kx - (M_1/P) - (R_1x/P)$$

The Macauley terms and expressions to be added to these basic equations are given in Table 1, those for bending moment being identical with the terms used for struts.

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